Journal of Engineering Mathematics, Vol. 10, No. 3, July 1976
Noordhoff International Publishing–Leyden
Printed in The Netherlands

An integral approach to lifting wing theory at Mach one*

T. R. GOODMAN

Oceanics, Inc., Technical Industrial Park, Plainview, New York 11803, *U.S.A.*

(Received August 15, 1975)

SUMMARY

An approach to lifting wing theory at Mach one is presented that utilizes an integral method similar to the Karman–Pohlhausen method in boundary layer theory. As in any integral method the results obtained are approximate in nature. Nonetheless, comparison with experimental data shows good agreement in cases for which experimental data are available. The method can easily be used to determine the lift on wings of finite aspect ratio and also to solve transient lifting problems. The method is demonstrated by solving for the pressure distribution on a lifting airfoil of arbitrary symmetric cross-section, the lift on a wing of rectangular planform, and the transient lift on an airfoil due to a sudden change in angle of attack. These cases were chosen to illustrate the versatility of the method and are not meant to be exhaustive of all possibilities. The computational time required to obtain numerical results is very small in all cases considered.

List of symbols

A	parameter associated with Guderley airfoil, defined in equation (28)
AR	aspect ratio
AR'	reduced aspect ratio $= AR\, \tau^{\frac{1}{3}}(\gamma + 1)^{\frac{1}{3}}$
c	chord of airfoil
C_l	sectional lift coefficient
C_L	lift coefficient
C_p	pressure coefficient
M	Mach number
p	Laplace transform variable
s	span of wing (in units of c)
t	time (in units of c/U)
U	free stream velocity
x	streamwise coordinate (in units of c)
x^*	distance from leading edge to sonic point (in units of c)
y	spanwise coordinate (in units of c)
z	coordinate normal to plane of wing (in units of c)
α	angle of attack
β	$\pi y/2s$
γ	ratio of specific heats ($= 1.4$ in all calculations)

* Research sponsored by the Air Force Office of Scientific Research/AFSC, United States Air Force, under Contract No. F44620-72-C-0079. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

δ	penetration depth (in units of c)
θ	defined in equation (13)
ξ	$x - \frac{2}{5}$
τ	thickness ratio of wing
ϕ	perturbation velocity potential (in units of $c \times U$)
ϕ_0	perturbation velocity potential associated with thickness
ϕ'	perturbation velocity potential associated with lift

1. Introduction

In light of the requirement for a transonic transport it is clearly of great importance to be able to predict the steady and unsteady lift on airfoils and wings of finite aspect ratio at Mach one. The difficulty of devising a theory capable of determining lift in the transonic speed regime can be attributed to the inherently nonlinear nature of the transonic small disturbance flow equation and also to the fact that the equation is of mixed elliptic-hyperbolic type. By considering the lift potential to be a small perturbation on the thickness potential, however, the transonic lift potential can be shown to satisfy a linear partial differential equation. Nevertheless, for $M \simeq 1$, this equation will still be of the mixed type, and difficulties encountered in attempting to solve it have hitherto proved to be insurmountable except in certain special cases. Finite difference schemes in conjunction with high-speed digital computers have been used to determine the flow about two-dimensional airfoils (see, e.g., [1]). However, the computation time may be great, and although the techniques may be extensible in principle to two- or three-dimensional steady or unsteady lifting problems, the amount of computation time required to solve such problems may prove to be exorbitant. It is therefore desirable to have a simple approximate method available for determining the solution to such problems. Recently, Stahara and Spreiter [2] have proposed an extension of the method of local linearization to determine the lift distribution on transonic oscillating airfoils. It is not possible at this time to assess the accuracy of that theory since no transonic oscillating airfoil data exists with which to compare it. On the other hand, in the steady case comparisons can be made with data presented in [3] and [10]. More recently, a slightly different approach to the method of local linearization for steady lifting airfoils at subsonic speeds has been presented by Subramanian and Balakrishnan [4], and fairly good agreement with data presented in [3] at $M \simeq .7$ is demonstrated.

This paper deals with a completely different approach to lifting transonic wing problems. The approach stems from the observation that at a free stream Mach number of one there is little upstream influence and as a consequence, an integral method, similar to the Karman–Pohlhausen method in boundary layer theory, can be used. The integral method reduces the problem to the solution of a boundary-value problem in the plane of the wing and the reduced problem can frequently be solved in a simple way. The method is applied here to solve three different lifting problems: 1) the lift on a two-dimensional symmetrical airfoil of arbitrary cross-section at a small angle of attack; 2) the lift on a finite aspect-ratio wing of rectangular planform having an arbitrary symmetric airfoil section; 3) the transient lift on a two-dimensional Guderley airfoil (defined below) due to a sudden change in angle of attack. These three cases serve merely to illustrate the method and the first two are used to verify the results of the method by comparison with appropriate experimental data. The method itself is completely general with regard to airfoil shape and planform shape; its basic limitations are that the angle of attack (and camber) must be small in comparison with the thickness ratio, that the Mach number must be close to one, and that the effects of viscosity are ignored.

2. The boundary value problem

The unsteady three-dimensional small disturbance transonic potential flow equation is

$$(1 - M^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = M^2(\gamma + 1)\phi_x\phi_{xx} + 2M^2\phi_{xt} + M^2\phi_{tt}. \tag{1}$$

It will be assumed either that shock waves are so weak that they are automatically accounted for by equation (1) or that all shock waves occur at the trailing edge of the wing. For $M \simeq 1$ equation (1) reduces to

$$\phi_{yy} + \phi_{zz} = (\gamma + 1)\phi_x\phi_{xx} + 2\phi_{xt} + \phi_{tt}. \tag{2}$$

This will be taken to be the fundamental equation. It will now be assumed that the velocity potential can be split into two parts; the first part, due to thickness, will be denoted by ϕ_0, while the second part, due to lift, will be denoted by ϕ'. Upon substituting

$$\phi = \phi_0(x, y, z) + \phi'(x, y, z, t) \tag{3}$$

and retaining only linear terms in ϕ' there is obtained

$$\phi_{0_{yy}} + \phi_{0_{zz}} = (\gamma + 1)\phi_{0_x}\phi_{0_{xx}}, \tag{4}$$

$$\phi'_{yy} + \phi'_{zz} = (\gamma + 1)[\phi_{0_x}\phi'_{xx} + \phi'_x\phi_{0_{xx}}] + 2\phi'_{xt} + \phi'_{tt}. \tag{5}$$

Equation (5) may also be written

$$\phi'_{yy} + \phi'_{zz} = (\gamma + 1)\frac{\partial}{\partial x}(\phi_{0_x}\phi'_x) + 2\phi'_{xt} + \phi'_{tt}. \tag{6}$$

It is assumed that the solution to the thickness problem is known, so that ϕ_0 is given. In that case, equation (5) (or (6)) is a linear partial differential equation for ϕ' whose solution is sought.

The boundary condition for the lift potential on the upper surface of the wing for all cases to be considered is

$$\phi'_z = \alpha, \quad z = 0 \tag{7}$$

where α is the instantaneous angle of attack. At infinity the perturbation velocity potential as well as the disturbance velocity components must vanish.

The pressure coefficient can be obtained from the unsteady linearized Bernoulli equation. For thin wings this becomes

$$C_p = -2[\phi'_x + \phi'_t]. \tag{8}$$

3. The lifting airfoil at Mach one

For a two-dimensional steady lifting airfoil equation (6) reduces to

$$\phi'_{zz} = (\gamma + 1)\frac{\partial}{\partial x}(\phi_{0_x}\phi'_x). \tag{9}$$

The integral method consists in assuming that the disturbance caused by the lifting foil penetrates into the flow only to a distance δ, called the penetration depth, and that for $z < \delta$ the potential ϕ' can be represented by a polynomial in z whose coefficients depend on x. Equation (9) is then multiplied by dz and integrated from $z = 0$ to $z = \infty$, resulting in

$$-\alpha = (\gamma + 1)\frac{d}{dx}\int_0^\delta \phi_{0_x}\phi'_x dz \tag{10}$$

where the boundary condition, equation (7), has been applied and it is assumed that ϕ' and the disturbance velocity components vanish beyond $z = \delta$. The solution that is sought will satisfy equation (10) rather than equation (9). The simplest polynomial for ϕ' to satisfy is a linear distribution, and, in this case, the one that satisfies the boundary conditions is

$$\begin{aligned} \phi' &= -\alpha(\delta - z), & 0 < z < \delta, \\ &= 0, & z \geqq \delta. \end{aligned} \tag{11}$$

The next step is to assume that the integral appearing in equation (10) can be evaluated sufficiently accurately if ϕ_{0_x} is taken to be its value at the surface of the foil or, for a thin airfoil, at $z = 0$. In this case, ϕ_{0_x} can be taken outside the integral. This will be a reasonable approximation provided the variation of ϕ_{0_x} is sufficiently slow over the interval $0 < z < \delta$. This will be confirmed subsequently.* Equation (10) then simplifies to

$$-\alpha = (\gamma + 1)\frac{d}{dx}\left(\phi_{0_x}(x)\frac{d\theta}{dx}\right) \tag{12}$$

where

$$\theta = \int_0^\delta \phi' dz \tag{13}$$

whence, by virtue of equation (11),

$$\theta = -\frac{\alpha\delta^2}{2}, \tag{14}$$

equation (12) can be integrated twice with respect to x and the constants of integration determined by specifying that θ must be regular at the point x^* at which ϕ_{0_x} vanishes and must vanish at the leading edge, $x = 0$. The point x^* is, of course, the sonic point on the airfoil as determined by the thickness distribution alone since the free stream velocity is

* This assumption is tantamount to assuming that the coefficients in equation (5) are independent of z, an approximation that has been shown to be valid to the first order near the airfoil [5].

sonic. The result is

$$-\frac{\theta}{\alpha} = \frac{\delta^2}{2} = \frac{1}{\gamma+1}\int_0^x \frac{\xi - x^*}{\phi_{0_x}(\xi)}\,d\xi \tag{15}$$

Upon utilizing equations (8) and (11), there is finally obtained

$$\frac{C_p(\gamma+1)^{\frac{1}{3}}\tau^{\frac{1}{3}}}{\alpha} = \frac{\sqrt{2}\left(\dfrac{x - x^*}{\bar{\phi}_{0_x}}\right)}{\left\{\displaystyle\int_0^x \frac{\xi - x^*}{\bar{\phi}_{0_x}}\,d\xi\right\}^{\frac{1}{2}}} \tag{16}$$

where

$$\phi_{0_x} = \frac{\tau^{\frac{2}{3}}}{(\gamma+1)^{\frac{1}{3}}}\,\bar{\phi}_{0_x}. \tag{17}$$

The lift (including both upper and lower surfaces of the foil) is seen from equations (8) and (11) to be

$$C_l = 4\alpha\delta(x = 1) \tag{18}$$

which, by virtue of equations (14) and (15), becomes

$$\frac{C_l(\gamma+1)^{\frac{1}{3}}\tau^{\frac{1}{3}}}{\alpha} = 4\sqrt{2}\left\{\int_0^1 \frac{(\xi - x^*)\,d\xi}{\bar{\phi}_{0_x}}\right\}^{\frac{1}{2}}. \tag{19}$$

It is to be noted that no Kutta condition is required in order to determine the lift uniquely. This is so because the rear part of the foil is in a supersonic flow field. The Kutta condition is replaced by the condition of regularity at the sonic point. Before continuing, a simplified version of the integral method will be introduced which is developed from equation (5) instead of equation (6). Since ϕ_{0_x} vanishes somewhere along the airfoil, namely at $x = x^*$, it will be small everywhere over the airfoil, and hence the term $\phi_{0_x}\phi'_{xx}$ might be expected to be small in comparison with the term $\phi'_x\phi_{0_{xx}}$. The analysis obtained by ignoring this term altogether will be termed the simplified integral method. In this form the partial differential equation can easily be seen to be of the diffusion type. Proceeding in the same way as before leads to a first order equation in θ that replaces equation (12). The result is

$$-\alpha = (\gamma+1)\phi_{0_{xx}}\frac{d\theta}{dx}. \tag{20}$$

The condition of regularity at the sonic point is no longer required, and the solution now becomes

$$-\frac{\theta}{\alpha} = \frac{\delta^2}{2} = \frac{1}{\gamma+1}\int_0^x \frac{d\xi}{\phi_{0_{xx}}(\xi)} \tag{21}$$

which replaces equation (15). The pressure coefficient and lift coefficient for the simplified integral method are identical to equations (16) and (19) respectively with $(x - x^*)/\bar{\phi}_{0_x}$ replaced by $1/\bar{\phi}_{0_{xx}}$. It can be seen that for a linear variation of ϕ_{0_x} the two expressions are identical, and so, the simplified integral method can be viewed as the same as the integral method with ϕ_{0_x} replaced by its tangent at the sonic point. But the simplified integral method also provides insight into the assumption that ϕ_{0_x} can be taken outside the integral in equation (10), for, according to equation (21),

$$\delta^2 = \frac{2}{\gamma+1}\int_0^x \frac{d\xi}{\phi_{0_{xx}}(\xi)}$$

and the multiplicative constant, 2, that appears is a direct result of the assumption that the profile is linear. On the other hand, it is shown in [6] that the thickness potential ϕ_0 can also be determined using an integral method, but that a quadratic profile is necessary, in which case the penetration depth for the thickness potential is identical to this expression except that the factor 2 is replaced by a factor 6. Hence, the penetration depth for the lift problem is $1/\sqrt{3}$ times the penetration depth for the thickness problem which means that over the interval $0 < z < \delta$, ϕ_{0_x} remains close to its surface value. The argument given here depends for its validity on the fact that the linear profile is appropriate for lift problems. It will be shown in the next section that this is the only profile that can be used to give the correct answer for a low aspect-ratio wing.

Two examples of the above results will now be presented. Using the method of local linearization Spreiter and Alksne [7] have shown that for a parabolic arc profile whose coordinates are given by

$$z = 2\tau(x - x^2) \tag{22}$$

The streamwise perturbation velocity due to the thickness is given by

$$\bar{\phi}_{0_x} = \left[\frac{12}{\pi}(\ln 4x - 8x + 8x^2 + \tfrac{3}{2})\right]^{\frac{1}{3}}. \tag{23}$$

The sonic point for this airfoil lies at $x^* = \frac{1}{4}$. A second class of airfoils, known as Guderley airfoils, have coordinates given by

$$z = \tfrac{25}{12}\sqrt{\tfrac{5}{3}}\tau(1 - x)x^{\frac{3}{2}}. \tag{24}$$

These airfoils are characterized by a constant surface pressure gradient near $M = 1$. The streamwise perturbation velocity due to thickness is given by

$$\bar{\phi}_{0_x} = \frac{125}{24}\left[\frac{9\pi}{50}\right]^{\frac{1}{3}}(x - \tfrac{2}{5}). \tag{25}$$

The sonic point for this airfoil lies at $x^* = \frac{2}{5}$. The lifting pressure distribution due to angle of attack at Mach one for a 6 percent thick parabolic arc airfoil is shown in Figure 1 using both the integral method and the simplified integral method. A comparison is also shown with the result obtained for the same airfoil using the method of local linearization [2]. Similarly, the pressure distribution due to angle of attack for the Guderley airfoil is shown in Figure 2. Since ϕ_{0_x} is linear for this airfoil the integral method and simplified integral method yield identical results. The pressure distribution for this airfoil using the method of local linearization [2] is also shown.

It should be noted that the pressures due to angle of attack as obtained according to the methods developed in this paper are consistantly greater than those obtained using the

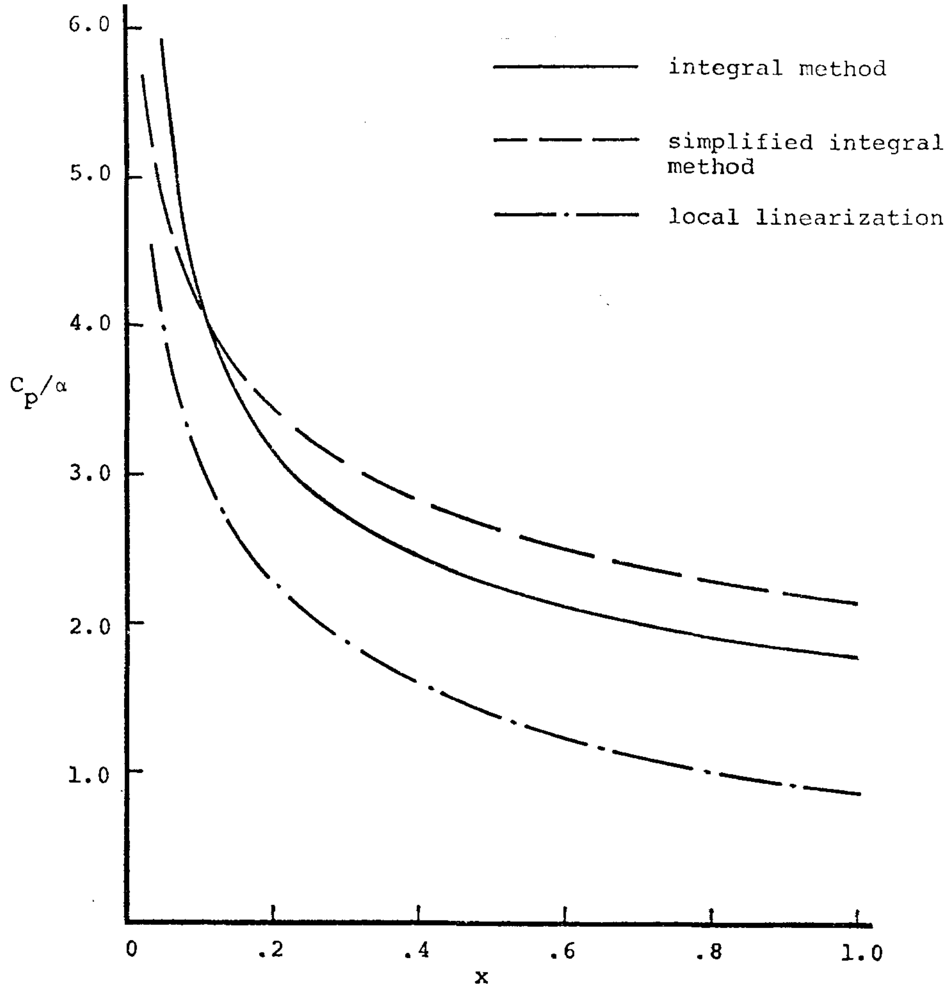

Figure 1. Pressure distribution on 6% thick parabolic arc airfoil—a comparison of three theories.

method of local linearization. In order to ascertain which of the methods yields superior results, a comparison with experiment is necessary. Such a comparison is shown in Figure 3 where the data points were obtained from [3] for a lifting circular arc airfoil in the following way: The pressure at zero angle of attack was subtracted from the pressure at 2° angle of attack for both upper and lower surfaces, this difference being the pressure due to angle of attack. Data points are for a free stream Mach number 1.007. The vertical lines through the data points indicate the accuracy with which the points can be read from the graphs presented in [3]. The corresponding theoretical curves were obtained from Figure 1 for a parabolic arc airfoil. (It should be noted that to the first order in thickness ratio a parabolic arc airfoil and a circular arc airfoil are the same.) From this comparison it can be seen that the simplified integral method gives the best fit to the data, the integral method gives an acceptable fit, but that the method of local linearization gives results that are consistantly low.

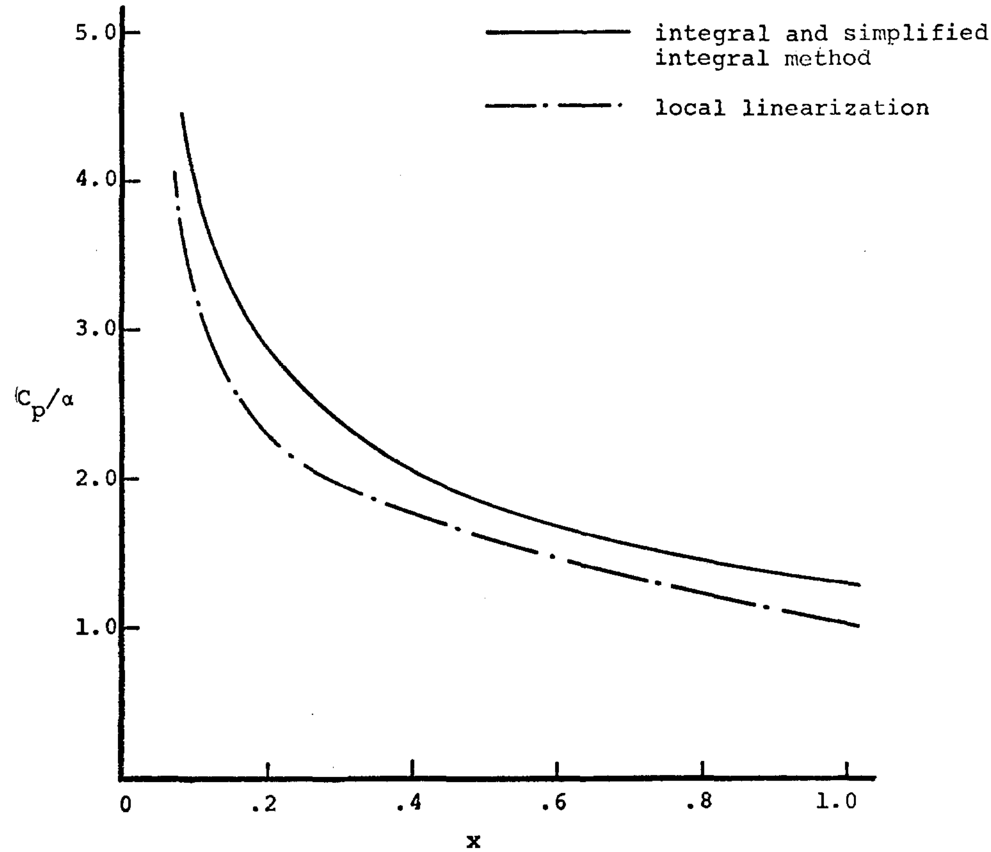

Figure 2. Pressure distribution on 6% thick Guderley airfoil—a comparison of three theories.

For the two airfoils considered the lift can be determined using the simplified integral equivalent of equation (19). For the parabolic arc profile this yields

$$C_l\tau^{\frac{1}{3}}(\gamma + 1)^{\frac{1}{3}}/\alpha = 3.08$$

while for the Guderley airfoil it yields

$$C_l\tau^{\frac{1}{3}}(\gamma + 1)^{\frac{1}{3}}/\alpha = 2.72.$$

For a 6% thick parabolic arc profile at $\alpha = 2°$ the theory yields $C_l = .205$ which agrees extremely well with the experimental values given in [3], namely $C_l = .195$ without a boundary layer trip and $C_l = .215$ with a boundary layer trip. For the same conditions the theoretical value of the moment coefficient about the midchord can be shown to be $C_{m.5} = .0148$, while the data presented in [3] indicates $C_{m.5} = .016$ without a trip and $C_{m.5} = .023$ with a trip. The difference in center of pressure location that comes about by using .0148 instead of .023 amounts to only 4% of the chord.

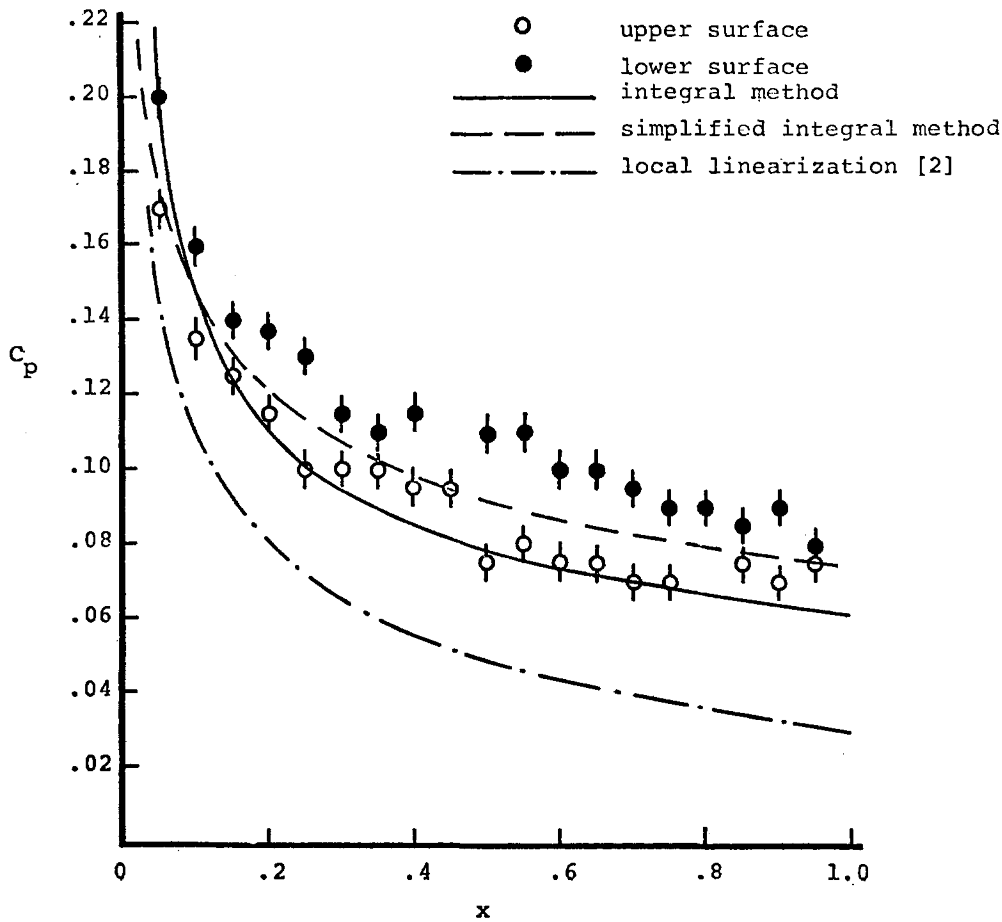

Figure 3. Comparison of experimental pressure distribution due to lift on 6% thick circular arc airfoil at 2° angle of attack with three theoretical distributions. (Note: Vertical lines through data points indicate accuracy to which data in [3] can be read.)

4. The lifting wing of finite aspect ratio at Mach one

Consider now a lifting wing of finite aspect ratio. For this case equation (6) reduces to

$$\phi'_{zz} + \phi'_{yy} = (\gamma + 1)\frac{\partial}{\partial x}(\phi_{0_x}\phi'_x). \tag{26}$$

Only a wing having a Guderley airfoil will be considered, in which case, upon integrating equation (26) with respect to z over the interval $0 < z < \delta$ and applying the boundary conditions there is obtained

$$-\alpha + \theta_{yy} = A\frac{d}{dx}[(x - \tfrac{2}{5})\theta_x] \tag{27}$$

where θ is defined by equation (13) and, from equations (17) and (25),

$$A = \tau^{\frac{2}{3}}(\gamma + 1)^{\frac{2}{3}} \frac{125}{24}\left[\frac{9\pi}{50}\right]^{\frac{1}{3}}. \tag{28}$$

Here the thickness velocity ϕ_{0_x} has been assumed to be determined using a strip theory so that the dependence of ϕ_{0_x} on y is ignored. The effect of this assumption will be discussed subsequently. Now, equation (27) is a partial differential equation which must be satisfied in the plane of the wing ($z = 0$) in the domain defined by the periphery of the wing. On the other hand, in the same plane but to port or starboard of the domain $\phi' = 0$; hence, according to equation (11), $\delta = 0$ outside the domain, and thus, by virtue of equation (14), $\theta = 0$. It is not necessary to consider the wake of the wing since the aft part of the wing is in a supersonic zone. Thus, for a wing of rectangular planform the boundary conditions that equation (27) must satisfy are:

$$\begin{aligned} &\theta(0, y) &&= 0 &&\text{(leading edge condition)} \\ &\theta(x, \pm s) &&= 0 &&\text{(tip condition)} \\ &\theta(\tfrac{2}{5}, y) &&= \text{regular} &&\text{(condition of regularity at sonic point)} \end{aligned} \tag{29}$$

Since it is only necessary to consider the domain of the wing, the constant angle of attack can be expanded in a Fourier series in the interval $-s < y < s$, in which case equation (27) becomes

$$-\frac{4\alpha}{\pi}\sum_{n=1,3}^{\infty}\frac{(-)^{(n-1)/2}}{n}\cos\left(\frac{n\pi y}{2s}\right) + \theta_{yy} = A\frac{d}{d\xi}\left(\xi\frac{\partial\theta}{\partial\xi}\right) \tag{30}$$

where

$$\xi = x - \tfrac{2}{5}. \tag{31}$$

Expanding θ in a Fourier series:

$$\theta = \sum_{n=1,3}^{\infty} f_n(\xi)\cos\frac{n\pi y}{2s}$$

which automatically satisfies the tip boundary conditions, the following ordinary differential equation for f_n is obtained:

$$A\frac{d}{d\xi}\left(\xi\frac{df_n}{d\xi}\right) + \left(\frac{n\pi}{2s}\right)^2 f_n = -\frac{4\alpha(-)^{(n-1)/2}}{n\pi}. \tag{32}$$

This can be recognized as a nonhomogeneous Bessel equation. The solution that satisfies the leading edge condition and the condition of regularity finally yields the following solution for θ:

$$\theta = \sum_{n=1,3}^{\infty}\frac{4\alpha}{n\pi}\left(\frac{2s}{n\pi}\right)^2(-)^{(n-1)/2}\cos\frac{n\pi y}{2s}\left\{\frac{I_0\left(\frac{n\pi}{s}\sqrt{\frac{-\xi}{A}}\right)}{I_0\left(\frac{n\pi}{s}\sqrt{\frac{2}{5A}}\right)} - 1\right\}, \tag{33a}$$

$-\frac{2}{5} < \xi < 0$ (upstream of sonic point) where I_0 is a modified Bessel function of the first kind, and

$$\theta = \sum_{n=1,3}^{\infty} \frac{4\alpha}{n\pi}\left(\frac{2s}{n\pi}\right)^2 (-)^{(n-1)/2} \cos\frac{n\pi y}{2s}\left\{\frac{J_0\left(\frac{n\pi}{s}\sqrt{\frac{\xi}{A}}\right)}{I_0\left(\frac{n\pi}{s}\sqrt{\frac{2}{5A}}\right)} - 1\right\}, \tag{33b}$$

$0 < \xi < \frac{3}{5}$ (downstream of sonic point)

where J_0 is a Bessel function of the first kind.

Equation (8), (11) and (14) may again be applied to obtain an expression for the pressure coefficient. This will not be given explicitly, but the results are shown in a three-dimensional diagram in Figure 4 for a wing having a six percent thick airfoil and an aspect ratio of 6. The sectional lift coefficient (accounting for both upper and lower surfaces of the wing) according to equation (8) and (11) is given by:

$$C_l = 4\phi(x = 1) = 4\alpha\delta(x = 1). \tag{34}$$

Hence, utilizing equation (33b) there is obtained for the total lift

$$\frac{C_L}{\alpha} = \frac{32s}{\pi}\int_0^{\frac{1}{2}\pi}\left\{\frac{2}{\pi}\sum_{n=1,3}^{\infty}\frac{(-)^{(n-1)/2}}{n^3}\cos n\beta\left[1 - \frac{J_0\left(\frac{n\pi}{s}\sqrt{\frac{3}{5A}}\right)}{I_0\left(\frac{n\pi}{s}\sqrt{\frac{2}{5A}}\right)}\right]\right\}^{\frac{1}{2}} d\beta \tag{35}$$

where $\beta = \pi y/2s$.

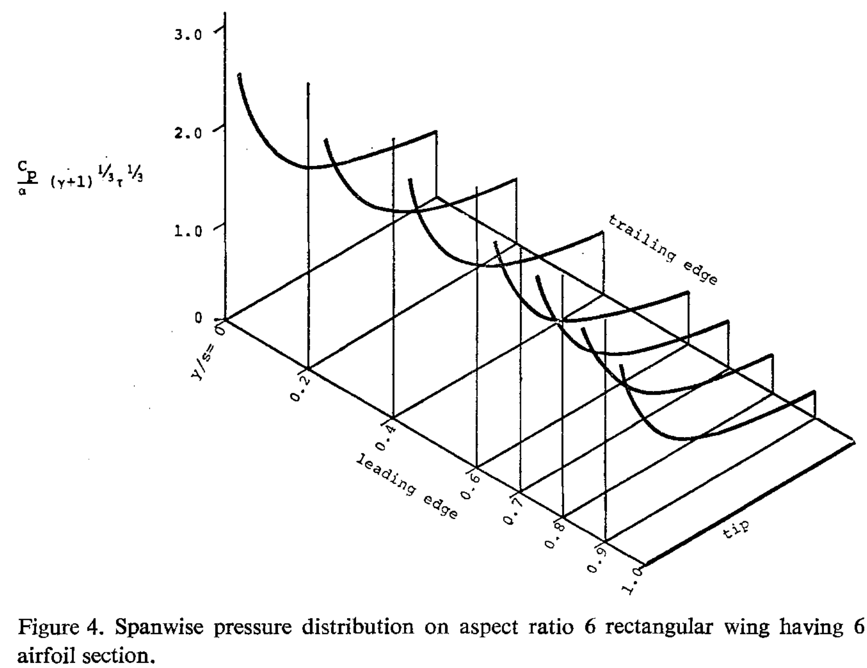

Figure 4. Spanwise pressure distribution on aspect ratio 6 rectangular wing having 6% thick Guderley airfoil section.

In the limit of small aspect ratio $s \to 0$, and the ratio of the Bessel functions approaches zero. The resulting series can be summed (see [8], series # 528) yielding,

$$\frac{C_L}{\alpha} \to \frac{32s}{\pi^2} \int_0^{\frac{1}{2}\pi} \left\{ \frac{1}{4} \left[\left(\frac{\pi^2}{2} \right) - \beta^2 \right] \right\}^{\frac{1}{2}} d\beta \tag{36}$$

which shows that the spanwise lift distribution is elliptic and independent of A. Furthermore, upon evaluating the integral there is obtained

$$\frac{C_L}{\alpha} \to s\pi \equiv \frac{\pi AR}{2} \tag{37}$$

which is the well-known result of slender-body theory.

Two points may now be made. In the first place, if a profile other than the linear profile had been used in setting up the integral method, the spanwise lift distribution would have still been elliptic in the limit of small aspect ratio, but the total lift would have differed from the correct slender-body result. Thus, the linear profile appears to be the most appropriate one to use. Secondly, as has been known for some time, the lifting properties of slender wings can be determined at Mach one using slender-body theory, and in this limit the lifting properties are independent of the thickness distribution of the airfoil. This is confirmed by equation (36) since the result is independent of A. Thus, for small aspect ratio wings it is not necessary to have any information at all concerning ϕ_{0x}, the streamwise perturbation velocity due to thickness, while for large aspect ratio wings it is essential to know ϕ_{0x}. But for large aspect ratio whings ϕ_{0x} is given, to a good approximation, by its two-dimensional value, and as the aspect ratio gets smaller and smaller the fact that a two-dimensional ϕ_{0x} is used becomes less and less important. This is the justification for using the two-dimensional value of ϕ_{0x} in setting up equation (27). Moreover, computations using a finite difference procedure indicate that ϕ_{0x} varies very little from the root to the tip of a rectangular wing [9].

Returning now to equation (35), the integrand represents the spanwise lift distribution, an example of which is shown in Figure 5 for a wing having a six percent thick airfoil and an aspect ratio of 6. Finally, similarity parameters can be introduced according to the following scheme:

$$\begin{aligned} &s \to AR/2, \;\; AR' \to AR\tau^{\frac{1}{3}}(\gamma+1)^{\frac{1}{3}}, \;\; A \to \frac{125}{24}\left(\frac{9\pi}{50}\right)^{\frac{1}{3}} \equiv 4.30698, \\ &\frac{C_L}{\alpha} \to \frac{C_L \tau^{\frac{1}{3}}(\gamma+1)^{\frac{1}{3}}}{\alpha}, \end{aligned} \tag{38}$$

The lift as a function of reduced aspect ratio is shown in Figure 6. It is seen that, as expected, the plot is tangent to the result obtained using slender-body theory for small aspect ratios and asymptotically approaches the result of two-dimensional theory at high aspect ratios.

The simplified integral method could have been used to solve the same problem. In that case, equation (27) would be replaced by

$$-\alpha = \theta_{yy} = A\frac{\partial \theta}{\partial x} \tag{39}$$

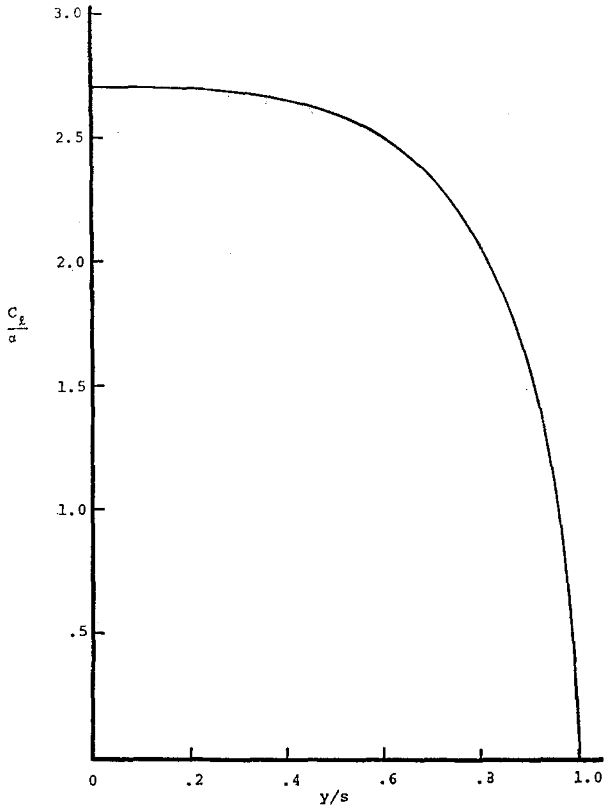

Figure 5. Spanwise sectional lift distribution on an aspect ratio 6 rectangular wing having 6% thick Guderley airfoil section.

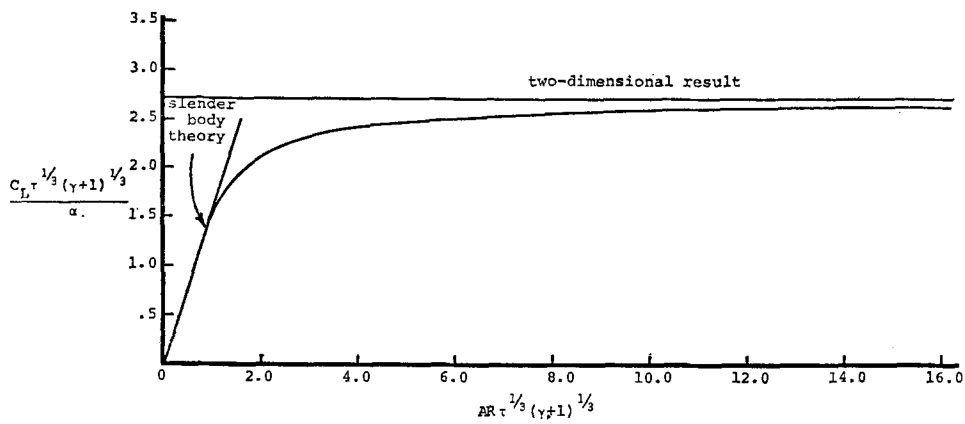

Figure 6. Lift coefficient as a function of reduced aspect ratio for a rectangular wing having a Guderley airfoil section.

and the method yields the following solution for θ:

$$\theta = \sum_{n=1,3}^{\infty} \frac{4\alpha}{n\pi}\left(\frac{2s}{n\pi}\right)^2 (-)^{(n-1)/2} \cos\frac{n\pi y}{2s}\left[\exp\left\{-\frac{1}{A}\left(\frac{n\pi}{2s}\right)^2 x\right\} - 1\right] \tag{40}$$

giving rise to the following expression for the total lift:

$$\frac{C_L}{\alpha} = \frac{32s}{\pi^2}\int_0^{\frac{1}{2}\pi}\left\{\frac{2}{\pi}\sum_{n=1,3}^{\infty}\frac{(-)^{(n-1)/2}}{n^3}\cos n\beta\left[1 - \exp\left\{-\frac{1}{A}\left(\frac{n\pi}{2s}\right)^2\right\}\right]\right\}^{\frac{1}{2}} d\beta \tag{41}$$

in place of equation (35). As is known, the simplified integral method yields results that are identical to the results obtained using the integral method for a two-dimensional Guderley airfoil. Both methods reduce to the result of slender-body theory at the low aspect ratio end. It has been ascertained by numerically evaluating equation (41) that the results obtained using either method cannot be distinguished for intermediate aspect ratios.

But the result of the simplified integral method is much more significant than at first appears because it is, in fact, the solution for a wing of rectangular planform having an arbitrary wing selection. This can be seen from equation (5) by ignoring the term $\phi_{0x}\phi'_{xx}$ and assuming $\phi_{0_{xx}}$ depends solely on x. Then upon letting

$$X = \frac{1}{(\gamma+1)}\int_0^x \frac{dx}{\phi_{0_{xx}}}. \tag{42}$$

Equation (5) reduces to

$$\phi'_{yy} + \phi'_{zz} = \frac{\partial \phi'}{\partial X}. \tag{43}$$

Hence, upon integrating over the interval $0 < z < \delta$, it is found that

$$-\alpha + \theta_{yy} = \frac{\partial \theta}{\partial X} \tag{44}$$

which is identical to equation (39) with x/A replaced by X. Thus the solution for a wing of rectangular planform having an arbitrary airfoil section is given by equation (40) with this replacement, and the total lift is given by equation (41) with $1/A$ replaced by

$$\frac{1}{(\gamma+1)}\int_0^1 \frac{dx}{\phi_{0_{xx}}}.$$

It can be seen that once $\phi_{0_{xx}}$ is specified for the airfoil, the lift on a finite wing of rectangular planform can immediately be determined.

This approach has been used to determine the lift distribution on an aspect ratio 3 rectangular wing having a 5% thick biconvex airfoil section at an angle of attack of 5°. Equation (23) was used for the sectional velocity. Then, upon combining equations (8), (14), and (40) with the replacement given above, sectional pressure distributions were calculated. Comparisons with the experimental data presented in [10] are shown in Figure 7 at four spanwise stations. It can be seen at a glance that the theoretical results at the 0.5, 0.7 and 0.9 spanwise stations agree extremely well with the data. The agreement at the 0.0 station (midspan) is not so good. However, the experiments were performed using a half

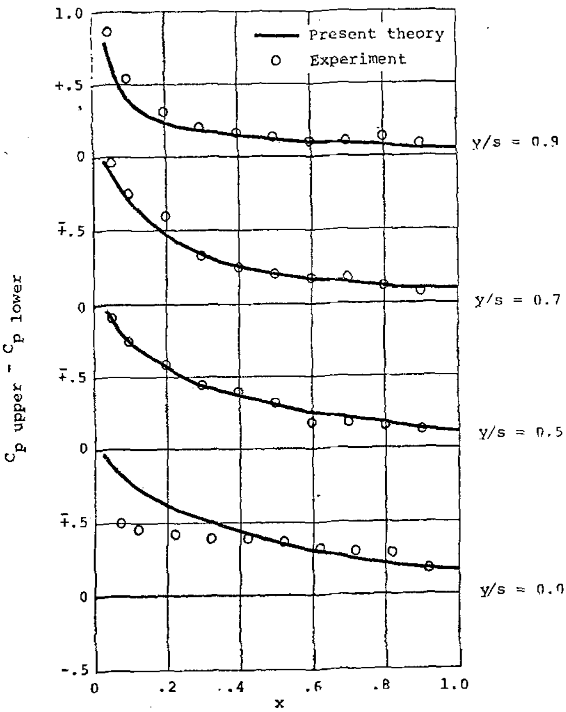

Figure 7. Comparison of experimental pressure distribution due to lift on rectangular wing of aspect ratio 3 having 5% thick biconvex airfoil section at 5° angle of attack with theoretical distribution at four spanwise stations. Experimental data from reference [10].

wing mounted on the wall of a wind tunnel and as a consequence the wall boundary layer is likely to have created interference causing the measured pressures at midspan to differ from the pressures on a full-span wing. Moreover, the pressure distribution at midspan should closely resemble a two-dimensional distribution, which the theoretical distribution does but not the experimental distribution. Because of these considerations it is likely that the theoretical distribution at midspan is more nearly correct than the experimental distribution.

It is worth noting that all the calculated data displayed on Figure 7 was generated in 1.6 seconds on the CDC 6600. No other method is known that can perform the required computations so rapidly. Furthermore, although the theory was developed under the assumption that the angle of attack is small in comparison with the thickness ratio, the case chosen for comparison violates this assumption and gives excellent agreement with the data nonetheless. This would seem to indicate that the theory is valid over a greater range of thickness ratios and angles of attack than might have been expected.

5. Transient lift due to sudden change in angle of attack at Mach one

The last problem that will be considered is the transient lift on a two-dimensional airfoil due to a sudden change in angle of attack at $M \simeq 1$. Such a solution is required in order to determine the gust response of the foil. Also, by Fourier analyzing this solution the lift on an oscillating airfoil may be determined for use in solving flutter problems. For this problem only the simplified integral method will be considered. Upon applying the method in the usual way, equation (5) becomes

$$-\alpha\, 1(t) = (\gamma + 1)\phi_{0_{xx}}\theta_x + 2\theta_{xt} + \theta_{tt} \tag{45}$$

where $1(t)$ is the unit step function. Upon taking the Laplace transform with respect to time, there is obtained

$$\frac{-\alpha}{p} = (\gamma + 1)\phi_{0_{xx}}\theta_x + 2p\theta_x + p^2\theta \tag{46}$$

where p is the Laplace transform variable. The solution that satisfies the condition $\theta(0) = 0$, is

$$\theta = -\frac{\alpha}{p^3}\left[1 - \exp\left\{-p^2\int_0^x \frac{dx}{(\gamma+1)\phi_{0_{xx}} + 2p}\right\}\right]. \tag{47}$$

For an arbitrary airfoil the indicated integration as well as the Laplace inversion would have to be carried out numerically. But for a Guderley airfoil both operations can be carried out analytically, and attention will be confined to this case. For the Guderley airfoil, $(\gamma + 1)\phi_{0_{xx}} \equiv A$, which is defined in equation (28), and equation (47) reduces to

$$\frac{\theta}{-\alpha} = \frac{\delta^2}{2} = \left\{1 - \exp\left(\frac{-p^2x/2}{A/2 + p}\right)\right\}\Big/ p^3 \tag{48}$$

Let

$$f(p; x) = \frac{\exp\left(\dfrac{-p^2x/2}{A/2 + p}\right)}{A/2 + p}. \tag{49}$$

Then, it can be shown, using tables of Laplace transforms (e.g., [11]), that $f(p; x)$ inverts to

$$\begin{aligned} f(t; x) &= e^{-A/2(t-x)} J_0\left[A\left\{\frac{x}{2}\left(t - \frac{x}{2}\right)\right\}^{\frac{1}{2}}\right], & t &\geqq \frac{x}{2}, \\ &= 0, & t &< \frac{x}{2}, \end{aligned} \tag{50}$$

where J_0 is a Bessel function of the first kind.

Since $1/p$ means integration, the solution can then be determined by solving the following set of ordinary differential equations:

$$\frac{dy_1}{dt} = \tfrac{1}{2}f(t; x), \quad \frac{dy_2}{dt} = 1 - f(t; x) - Ay_1, \quad \frac{dy_3}{dt} = y_2 \tag{51}$$

with initial conditions $y_1(0, x) = y_2(0, x) = y_3(0, x) = 0$.

By integrating over the infinite interval it is not difficult to show, using a table of Laplace transforms, that in the limit as $t \to \infty$, $y_1 \to 1/A$, $y_2 \to 0$, $y_3 \to x/A$. Furthermore, it is possible to establish the following identifications:

$$y_1 \equiv \frac{\partial(\theta/-\alpha)}{\partial x}, \; y_2 \equiv \frac{\partial(\theta/-\alpha)}{\partial t}, \; y_3 \equiv (\theta/-\alpha). \tag{52}$$

Thus, the pressure coefficient becomes

$$C_p/\alpha = \sqrt{\frac{2}{y_3}}\,[y_1 + y_2] \tag{53}$$

while the lift (including upper and lower surfaces of the foil) becomes

$$\frac{C_l}{\alpha} = 2\sqrt{2}\left[2[y_3(x=1)]^{\frac{1}{2}} + \int_0^1 \frac{y_2}{(y_3)^{\frac{1}{2}}}\,dx\right]. \tag{54}$$

Now it is easy to see that for $t < x/2$

$$y_1 = 0, \; y_2 = t, \; y_3 = t^2/2. \tag{55}$$

Hence, at $t = 0$ the pressure distribution is uniform and the lift instantaneously jumps to the value*

$$\frac{C_l}{\alpha}(t=0) = 4. \tag{56}$$

An attempt was made to integrate equations (51) numerically; however, it was found that because there is no inherent feedback in the equations it was impossible to drive y_2 to zero for large time, and, instead, y_2 appeared to drift upward at a rate that depended on the integration step size. The problem was resolved by casting equation (50) into a form such that the integrations could be performed analytically. By using the identity,

$$(xz)^{-\frac{1}{2}a} e^z J_a[2(xz)^{\frac{1}{2}}] = \sum_{n=0}^{\infty} \frac{L_n^a(x) z^n}{\Gamma(n+a+1)} \tag{57}$$

which is given in Section 10.12 of [12], the function $f(t; x)$ can be shown to be represented by an infinite sum:

$$\begin{aligned} f(t;x) &= e^{-A/2(t-x/2)} \sum_{n=0}^{\infty} \frac{L_n^0\left[\frac{A}{2}\left(t-\frac{x}{2}\right)\right]\left(\frac{Ax}{4}\right)^n}{n!}, && t \geqq \frac{x}{2}, \\ &= 0, && t < \frac{x}{2}. \end{aligned} \tag{58}$$

Here $L_n^a(x)$ is a generalized Laguerre polynomial that satisfies the recursion relationship

$$(n+1)L_{n+1}^a(x) - (2n+a+1-x)L_n^a(x) + (n+a)L_{n-1}^a(x) = 0 \tag{59}$$

* This result is identical with the initial lift and pressure distribution obtained using linearized theory at both subsonic and supersonic speeds.

with the initial conditions

$$L_0^a(x) = 1, \ \ L_1^a(x) = a + 1 - x. \tag{60}$$

By integrating over the infinite interval, making use of the limiting results, and subtracting away the integral from t to infinity it can be shown that

$$\begin{aligned} y_1 &= \frac{1}{A}\left\{1 - e^{-A/2(t-x/2)} \sum_{n=0}^{\infty} \left(\frac{Ax}{4}\right)^n \frac{L_n^{-1}\left[\frac{A}{2}\left(t - \frac{x}{2}\right)\right]}{n!}\right\}, \\ y_2 &= \frac{2}{A}\, e^{-A/2(t-x/2)} \sum_{n=1}^{\infty} \left(\frac{Ax}{4}\right)^n \frac{L_{n-1}^{-1}\left[\frac{A}{2}\left(t - \frac{x}{2}\right)\right]}{n!}, \\ y_3 &= \frac{x}{A}\left\{1 - e^{-A/2(t-x/2)} \sum_{n=1}^{\infty} \left(\frac{Ax}{4}\right)^{n-1} \frac{L_{n-1}^{-2}\left[\frac{A}{2}\left(t - \frac{x}{2}\right)\right]}{n!}\right\}, \qquad t > \frac{x}{2}, \end{aligned} \tag{61}$$

where use has been made of the identities, given in [12],

$$\int_x^{\infty} e^{-y} L_n^a(y)\, dy = e^{-x}[L_n^a(x) - L_{n-1}^a(x)], \tag{62}$$

$$L_n^{a-1}(x) = L_n^a(x) - L_{n-1}^a(x). \tag{63}$$

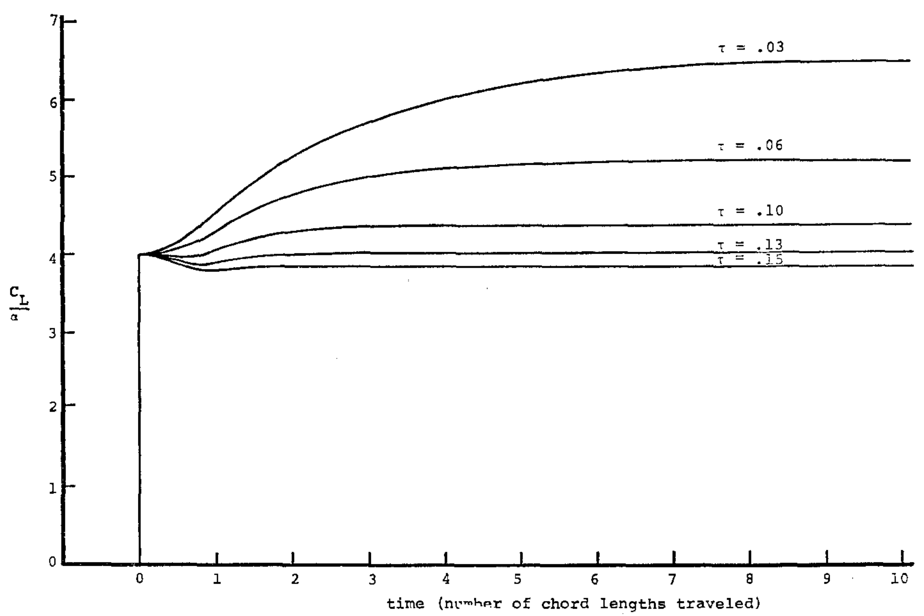

Figure 8. Transient lift response of a Guderley airfoil due to a sudden change in angle of attack.

Numerical evaluation shows that the series converge very rapidly for all values of $t \geqq x/2$. The lift response was calculated using equation (54), where the integral was evaluated using the trapezoid rule. Results are shown in Figure 8. It is interesting to observe that the lift coefficient initially jumps to 4α, and eventually reaches $4\alpha(2/A)^{\frac{1}{2}}$. Thus, for $A = 2$ the initial and final values of the lift are identical though the pressure distributions are completely different. This case corresponds to a thickness ratio of approximately $\tau = .13$. Of course, thickness ratios greater than 13% are not likely to be encountered in practice. Transient lift growth is illustrated in Figure 8 for several values of τ. It can be seen that the time for C_l/α to reach its steady state value is very short for the thicker foils, being less than the time to travel two chord lengths for $\tau = .15$.

6. Conclusions

An integral approach to lifting wing theory at Mach number one has been developed and has been applied to solve three problems: 1) the lift on an airfoil section of arbitrary cross section at angle of attack; 2) the lift on a wing of rectangular planform at angle of attack; 3) the transient lift on a Guderley airfoil due to a sudden change in angle of attack. For the first and second problems some experimental data is available and agrees well with the theory. For the third problem no experimental data seems to be available with which to make comparisons. All computations were carried out on a CDC 6600. The amount of computation time required in all cases was a few seconds, a feature of the present approach that is certainly of some significance.

REFERENCES

[1] E. M. Murman and J. D. Cole, Calculation of plane steady transonic flows, *AIAA Journal* 9 (1971) 114–121.
[2] S. S. Stahara and J. R. Spreiter, Development of a non-linear unsteady transonic flow theory, *NASA Contractor Report* 2258 (1973).
[3] E. D. Knechtel, Experimental investigation at transonic speeds of pressure distributions over wedges and circular arc airfoil sections and evaluation of perforated wall interference, *NASA TN D*-15 (1959).
[4] N. R. Subramanian and M. Balakrishnan, Transonic flow past lifting airfoils, *AIAA Journal* 2 (1973) 1766–1768.
[5] P. E. Ruppert and M. T. Landahl, Solution of the transonic airfoil problem through parametric differentiation, *AIAA Journal* 5 (1967) 470–479.
[6] T. R. Goodman, Wall interference on airfoils in transonic tunnels at Mach one, *AFOSR-TR*-73-0492, *Oceanics Report No.* 73–95 (1973).
[7] J. R. Spreiter and A. Y. Alksne, Thin airfoil theory based on approximate solution of the transonic flow equation, *NACA TR* 1359 (1958).
[8] L. B. W. Jolley, *Summation of Series*, Second Revised Edition, Dover Publications, New York (1961).
[9] P. K. Khosla and S. G. Rubin, Transonic flow calculations in two- and three-dimensions, *AFOSR TR*-73-1976, *PIBAL Report* 73-13 (1973).
[10] H. C. Lessing, J. L. Troutman and G. P. Meness, Experimental determination of the pressure distribution on a rectangular wing oscillating in the first bending mode for Mach Numbers from 0.24 to 1.30, *NASA TN D*-344 (1960).
[11] A. Erdelyi, Ed., *Tables of Integral Transforms—Vol. I*, McGraw-Hill Book Company, New York (1954).
[12] A. Erdelyi, Ed., *Higher Transcendental Functions—Vol. II*, McGraw-Hill Book Company, New York (1953).